Optimal contraction of the energy difference for strongly monotone problems

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Motivation

Problem

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Convex energy minimization problem

$$u = \operatorname*{arg\,min}_{v \in H_0^1(\Omega)} \mathcal{J}(v), \qquad \mathcal{J}(v) := \int_\Omega \phi(|\nabla v|) - f\,v\,\mathrm{d}\boldsymbol{x}. \tag{1}$$

- $\phi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ convex
- $\phi(0) = 0$, $\phi'(0) = 0$
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Equivalently, $u \in H_0^1(\Omega)$ solves the Euler-Lagrange equation

$$(a(|\nabla u|)\nabla u, \nabla v)_\Omega = (f, v)_\Omega \qquad \text{for all } v \in H_0^1(\Omega). \tag{2}$$

with nonlinear *effective diffusivity* $a(s) := \phi'(|s|)/|s|$ and nonlinear *flux* $\boldsymbol{A}(\nabla u) := a(|\nabla u|)\nabla u$.

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Example ("Kink-nonlinearity")

Define the potential for $a_c \geq a_m > 0$ via

$$\phi'(0) = 0, \qquad \phi''(s) := \begin{cases} a_m, & s \leq s_0, \\ a_c, & \text{else}. \end{cases} \tag{2}$$

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ANR & DFG project RANPDEs

- Perform analysis in terms of the *energy difference* $\mathcal{J}(u_\ell) - \mathcal{J}(u)$ (done for H1-seminorm error *e.g.* in [BBDL, '25], [KV, '19], [ESV, '16], [BPS, '09], for norm errors in general *e.g.* in [HW, '20]), and for energy difference in a quasi-linear case *e.g.* in [HPW, '21]

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Localized energy-based perspective, with focus on the *a posteriori* error estimators

Part I: Abstract conditions for optimal contraction

Notation

- $\Omega \subset \mathbb{R}^d$: bounded Lipschitz domain
- $\mathcal{T}$: simplicial and conforming triangulation
- $\mathcal{V}_\ell$: set of vertices on mesh level ℓ
- $V^p(\mathcal{T}_\ell)$: piecewise polynomials of degree p on the mesh $\mathcal{T}_\ell$
- $\mathbb{T}$: collection of all possible conforming refinements of the initial triangulation $\mathcal{T}_0$ obtained by iterative bisection and subsequent conforming closure.
- $\mathbb{T}_N \subseteq \mathbb{T}$: set of all refinements which have at most N more elements compared to the initial triangulation $\mathcal{T}_0$

Energy-norm equivalence

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Usually: if we "pay" the constants a_m / a_c, we could then reuse (parts of) the analysis of the quadratic problem as

$$\frac{a_m}{2}\|\nabla(u_\ell - u)\| \leq \mathcal{J}(u_\ell) - \mathcal{J}(u) \leq \frac{a_c}{2}\|\nabla(u_\ell - u)\| \tag{3}$$

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Lemma (Relation to norms)

Let $u \in H_0^1(\Omega)$ be the minimizer and $v \in H_0^1(\Omega)$ arbitrary. Then, there holds

$$\mathcal{J}(v) - \mathcal{J}(u) = \tfrac{1}{2}\|\rho(\nabla v, -\boldsymbol{A}(\nabla u))\nabla(u - v)\|^2. \tag{4}$$

The weight ρ is globally bounded as $a_m \leq \rho \leq a_c$.

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Similar equality holds with a weighted flux difference, as well as for the primal-dual gap $\mathcal{J}(u) - \mathcal{J}^*(v)$
Consequence: (4) is a powerful equality (delivers at least *localization*), provided we can estimate ρ well locally

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Lemma (Pointwise bounds for ρ)

It holds that

$$\min\{\lambda(|\boldsymbol{x}|), \Lambda(|\boldsymbol{x}|, |\boldsymbol{y}|)\} \leq \rho(\boldsymbol{x}, \boldsymbol{y}) \leq \max\{\lambda(|\boldsymbol{x}|), \Lambda(|\boldsymbol{x}|, |\boldsymbol{y}|)\}, \tag{5}$$

with

$$\lambda(r) := 1/a^*(r), \qquad \Lambda(s, r) := \int_0^1 2(1-\tau)\phi''(\tau s + (1-\tau)(\phi^*)'(r))\, d\tau \tag{6}$$

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Adaptive loop
Algorithm

Standard AFEM-loop

$$\texttt{SOLVE} \to \texttt{ESTIMATE} \to \texttt{MARK} \to \texttt{REFINE} \tag{AFEM}$$

with the components

- SOLVE: Compute the discrete minimizer u_ℓ on the current refinement level ℓ by solving the corresponding weak formulation.
- ESTIMATE: Compute the local indicators $\overline{\eta}_\ell^{\boldsymbol{a}}$ from the current approximation u_ℓ.
- MARK: Based on the local indicators $\overline{\eta}_\ell^{\boldsymbol{a}}$, determine a set $\mathcal{M}_\ell \subset \mathcal{V}_\ell$ fulfilling the marking criterion [Dörfler, '96]

$$\sum_{\boldsymbol{a}\in\mathcal{M}_\ell} \overline{C}_{\mathrm{loc}}^{\boldsymbol{a}} \overline{\eta}_\ell^{\boldsymbol{a}} \geq \theta \sum_{\boldsymbol{a}\in\mathcal{V}_\ell} \overline{C}_{\mathrm{loc}}^{\boldsymbol{a}} \overline{\eta}_\ell^{\boldsymbol{a}}. \tag{11}$$

  where $0 < \theta \leq 1$ is a fixed **marking parameter**.
- REFINE: Return a new mesh $\mathcal{T}_{\ell+1} := \texttt{REFINE}(\mathcal{T}_\ell, \mathcal{M}_\ell) \in \mathbb{T}$ which is a refinement of $\mathcal{T}_\ell$ by newest vertex bisection (potentially multiple times) + conforming closure.

Contraction

Theorem and insights into part II

Theorem (Contraction)

Let $(u_\ell)_{\ell \in \mathbb{N}}$ be the sequence of approximations generated by the adaptive loop and assume the conditions 1. to 4. hold. Then the energy difference contracts in each iteration, i.e. for any $\ell \in \mathbb{N}$ with $u_\ell \neq u$ it holds that

$$\frac{\mathcal{J}(u_{\ell+1}) - \mathcal{J}(u)}{\mathcal{J}(u_\ell) - \mathcal{J}(u)} < 1$$

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However: shows better behavior for smooth nonlinearities (see concrete illustration in Part II)

Optimality
Approximation class

Definition (Approximation class)

For any $s > 0$ we set

$$\|u\|_{\mathbb{A}_s} := \sup_{N \in \mathbb{N}_0} \min_{\mathcal{T}_\star \in \mathbb{T}_N} \left\{ (N+1)^s \left(\mathcal{J}(u_{\mathcal{T}_\star}) - \mathcal{J}(u) \right) \right\} \tag{12}$$

and say that the solution u is an element of the approximation class $\mathbb{A}_s$ if and only if $\|u\|_{\mathbb{A}_s}$ is finite (i.e. if the convergence rate s is attainable by an optimal sequence of meshes).

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Additional assumptions and theorem

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6. **Discrete reliability:** For any mesh $\mathcal{T}_\ell$, let $\mathcal{S}_\ell \subset \mathcal{V}_\ell$ be a marked set and define $\widehat{\mathcal{T}}_\ell := \mathtt{REFINE}(\mathcal{T}_\ell, \mathcal{S}_\ell)$. With u_ℓ, $\hat{u}_\ell$ the discrete minimizers on the respective meshes, it holds that

$$\mathcal{J}(u_\ell) - \mathcal{J}(\hat{u}_\ell) \leq \sum_{\boldsymbol{a} \in \mathcal{V}_\ell^1(\mathcal{S}_\ell)} C_{\mathrm{drel}}^{\boldsymbol{a}} \, \overline{C}_{\mathrm{loc}}^{\boldsymbol{a}} \, \overline{\eta}_\ell^{\boldsymbol{a}}. \tag{13}$$

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$$\mathcal{J}(u_\ell) - \mathcal{J}(\hat{u}_\ell) \leq \sum_{\boldsymbol{a} \in \mathcal{V}^1_\ell(\mathcal{S}_\ell)} C^{\boldsymbol{a}}_{\mathrm{drel}} \, \overline{C}^{\boldsymbol{a}}_{\mathrm{loc}} \, \overline{\eta}^{\boldsymbol{a}}_\ell. \tag{13}$$

Then there holds (following the lines of the proof in [CFPP, '14]):

Theorem (Optimal convergence)

Let the marking parameter $\theta \leq \theta_{\mathrm{opt}}$ be sufficiently small and let $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}}$ be the sequence of meshes generated by the adaptive algorithm (AFEM). Then the energy difference of the corresponding approximations $(u_\ell)_{\ell \in \mathbb{N}}$ converges with the optimal rate w.r.t. #DOFs, i.e. for any $s > 0$ there holds

$$c_{\mathrm{opt}} \, \|u\|_{\mathbb{A}_s} \leq \sup_{\ell \in \mathbb{N}} \{(\#\mathcal{T}_\ell - \#\mathcal{T}_0 + 1)^s (\mathcal{J}(u_{\mathcal{T}_\star}) - \mathcal{J}(u))\} \leq C_{\mathrm{opt}} \, \|u\|_{\mathbb{A}_s}. \tag{14}$$

Conclusion / outlook

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By [GHPS, '21], this would also imply optimality w.r.t. the total *computational cost*

Part II: Application to flux, residual and energy descent estimators

Overview

Part I: *Optimal contraction with given upper- and lower bound estimators*

$\rightarrow$ **Part II:** How do established a posteriori estimators fit into this framework?
What about the *localized* constants?

We will discuss:

- An upper bound $\overline{\eta}_\ell := \eta_{\text{flux},\ell}$ using *flux-equilibration (linear)*
- A lower bound $\underline{\eta}_\ell := \eta_{\text{energy},\ell}$ using *local energy minimization (nonlinear)*
- Linear local *residual liftings* $\eta_{\text{reslift},\ell}$
- A *residual* upper bound $\eta_{\text{res},\ell}$ *(element- and face residuals)*

All localizations will be based on *vertex-patches* $\omega_{\boldsymbol{a}}$!

Locally, there holds (under certain assumptions, neglecting data oscillation)

$$\overline{\eta}_\ell^{\boldsymbol{a}} := \eta_{\text{flux},\ell}^{\boldsymbol{a}} \lesssim \eta_{\text{res},\ell}^{\boldsymbol{a}} \lesssim \eta_{\text{reslift},\ell}^{\boldsymbol{a}} \lesssim \eta_{\text{energy},\ell}^{\boldsymbol{a}} =: \underline{\eta}_\ell^{\boldsymbol{a}}.$$

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$\rightarrow$ **Part II:** How do established a posteriori estimators fit into this framework?
What about the *localized* constants?

We will discuss:

- An upper bound $\overline{\eta}_\ell := \eta_{\mathrm{flux},\ell}$ using *flux-equilibration (linear)*
- A lower bound $\underline{\eta}_\ell := \eta_{\mathrm{energy},\ell}$ using *local energy minimization (nonlinear)*
- Linear local *residual liftings* $\eta_{\mathrm{reslift},\ell}$
- A *residual* upper bound $\eta_{\mathrm{res},\ell}$ *(element- and face residuals)*

All localizations will be based on *vertex-patches* $\omega_{\mathbf{a}}$!

Locally, there holds (under certain assumptions, neglecting data oscillation)

$$\overline{\eta}_\ell^{\mathbf{a}} := \eta_{\mathrm{flux},\ell}^{\mathbf{a}} \lesssim \eta_{\mathrm{res},\ell}^{\mathbf{a}} \lesssim \eta_{\mathrm{reslift},\ell}^{\mathbf{a}} \lesssim \eta_{\mathrm{energy},\ell}^{\mathbf{a}} =: \underline{\eta}_\ell^{\mathbf{a}}.$$

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Flux-based upper bound $\eta_{\text{flux},\ell}$

Idea: Construct a dual object $\boldsymbol{\sigma}_\ell \in \boldsymbol{H}(\text{div}, \Omega)$, $\nabla\cdot\boldsymbol{\sigma}_\ell = \Pi_q f$ from $u_\ell \in V_\ell^p$

Ansatz: Linear equilibration of nonlinear flux [HMRV, '24] (here also allowing for higher degree $q \geq p$)

$$\boldsymbol{\sigma}_\ell := \sum_{\boldsymbol{a}\in\mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\boldsymbol{a}}, \qquad \boldsymbol{\sigma}_\ell^{\boldsymbol{a}} := \underset{\substack{\boldsymbol{w}_\ell \in \text{RT}_q(\mathcal{T}_\ell^{\boldsymbol{a}}) \cap \boldsymbol{H}(\text{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{w}_\ell = \Pi_q(\psi^{\boldsymbol{a}} f - \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{A}(\nabla u_\ell))}}{\arg\min} \|\psi^{\boldsymbol{a}}\Pi_{q-1}^{\text{RT}}(\boldsymbol{A}(\nabla u_\ell)) + \boldsymbol{w}_\ell\|_{\omega_{\boldsymbol{a}}} \tag{15}$$

- *Linear, homogeneous* local Neumann-problem with divergence constraint
- One has the localized upper bound (neglecting data oscillation)

$$\mathcal{J}(u_\ell) - \mathcal{J}(u) = \mathcal{J}(u_\ell) - \mathcal{J}^*(\boldsymbol{\sigma}) \qquad \textit{(strong convex duality / "Prager-Synge")}$$

$$\leq \mathcal{J}(u_\ell) - \mathcal{J}^*(\boldsymbol{\sigma}_\ell) \qquad (=: \overline{\eta}_\ell, \textit{ computable})$$

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A localized lower bound estimator $\underline{\eta}_\ell$

Local energy minimization

Task: Given the refinement $\mathcal{T}_\ell \to \mathcal{T}_{\ell+1}$, estimate $\mathcal{J}(u_\ell) - \mathcal{J}(u_{\ell+1})$ from below

$\to$ Compute lifting $\tilde{u}_{\ell+1} \approx u_{\ell+1}$ of u_ℓ:

$$\mathcal{J}(u_\ell) - \mathcal{J}(u_{\ell+1}) \quad \geq \quad \mathcal{J}(u_\ell) - \mathcal{J}(\tilde{u}_{\ell+1}) \qquad\qquad (u_{\ell+1} \text{ minimizer}).$$

Ansatz: Local **state liftings** $\tilde{u}^a_{\ell+1} \in \mathcal{P}_p(\mathcal{T}^a_{\ell,\ell+1}) \cap H^1(\omega_a) =: V^p_{\ell,\ell+1}$

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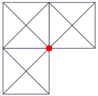
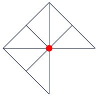
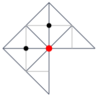


Domain Ω / mesh $\mathcal{T}_\ell$



Subdomain $\omega_{\boldsymbol{a}}$ / mesh $\mathcal{T}^{\boldsymbol{a}}_\ell$



Local refinement $\mathcal{T}^{\boldsymbol{a}}_{\ell,\ell+1}$ (1 NVB)

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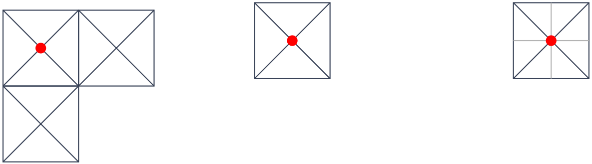
Discrete efficiency
Sufficient / practical refinement conditions

Reminder: Contraction requires localized upper/lower bounds and *discrete efficiency* $\overline{\eta}_\ell^{\boldsymbol{a}} \leq C_{\text{deff}}^{\boldsymbol{a}} \underline{\eta}_\ell^{\boldsymbol{a}}$.

Observation: This poses a requirement on the amount of refinement



Insufficient local refinement, no new degrees of freedom introduced, $\underline{\eta}_\ell^{\boldsymbol{a}} = 0$

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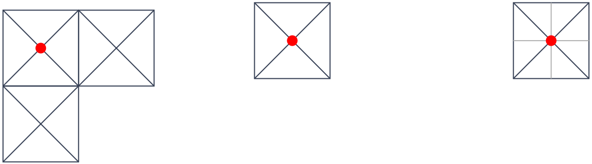

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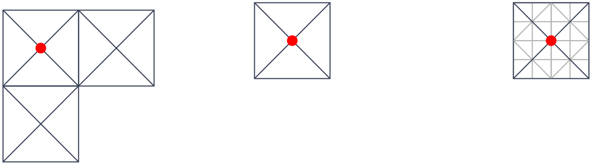

Sufficient local refinement, INP fulfilled, theoretically guaranteed disc. eff.

Interior node property (INP): The refinement must introduce an interior node on each element and on each internal face

- Allows for theoretical proof of discrete efficiency (based on discrete bubble functions)
- For our estimators: $C_{\mathrm{deff}}^{\boldsymbol{a}} = C(d, \kappa_{\mathcal{T}}, p) \left(\inf_{x \in \omega_{\boldsymbol{a}}} \{\tilde{\rho}(\nabla u_\ell, \nabla \tilde{u}_{\ell+1})\}\right)^{-1}$

INP can be avoided by an *adaptive approach* [Chaumont-Frelet, Dong, Gantner, Vohralík, '26]

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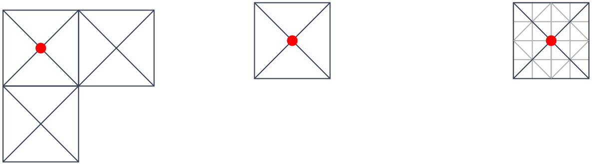

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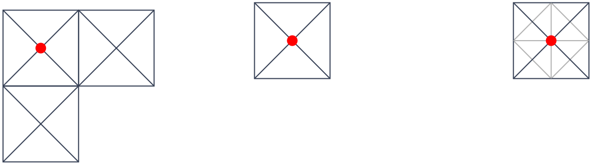

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Locality of constants

Contraction bound using flux + energy estimator

Reminder: Abstract contraction bound

$$\frac{\mathcal{J}(u_{\ell+1}) - \mathcal{J}(u)}{\mathcal{J}(u_\ell) - \mathcal{J}(u)} \leq 1 - \frac{\underline{\eta}_\ell}{\overline{\eta}_\ell} \leq 1 - \theta \left(\max_{a \in \mathcal{M}_\ell} \left\{ \frac{\overline{C}^a_{\mathrm{loc}}\, C^a_{\mathrm{deff}}}{\underline{C}^a_{\mathrm{loc}}} \right\} \right)^{-1}$$

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Bounded by computable local nonlinearity ratio *(visualization on next slide)*

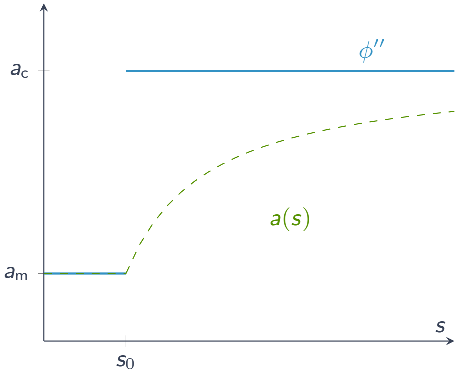
$$\mathfrak{R}_{\mathrm{nl}}(\omega_a) := \frac{\sup_{s \in I^a} \max\{\phi''(s), a(s)\}}{\inf_{s \in I^a} \min\{\phi''(s), a(s)\}}$$

with $I^a \subset \mathbb{R}_0^+$ given by
$I^a := |\nabla u_\ell|(\omega_a) \cup |\nabla \tilde{u}_{\ell+1}|(\omega_a) \cup |\boldsymbol{A}^{-1}\boldsymbol{\sigma}_\ell|(\omega_a)$

Locality of constants
Visualization for kink-nonlinearity

Question: How does the local ratio $\mathfrak{R}_{\mathrm{nl}}(\omega_{\boldsymbol{a}})$ behave?

Here: $\mathfrak{R}_{\mathrm{nl}}(\omega_{\boldsymbol{a}}) = \max_{s \in I^{\boldsymbol{a}}} \frac{\phi''(s)}{a(s)}$



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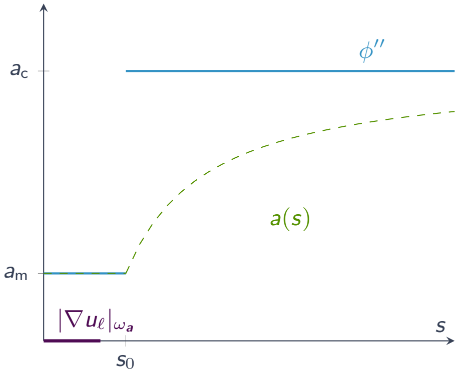

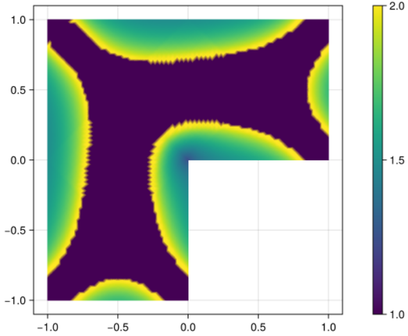

Local contrast (kink with $a_{\mathrm{m}} = 1$, $a_{\mathrm{c}} = 2$)

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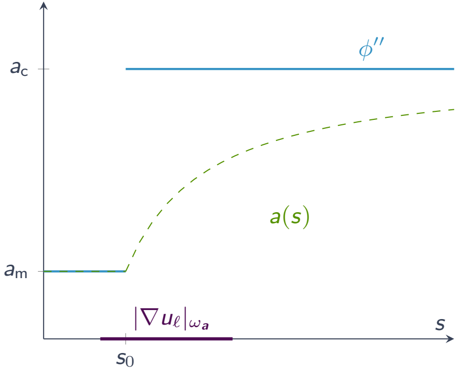

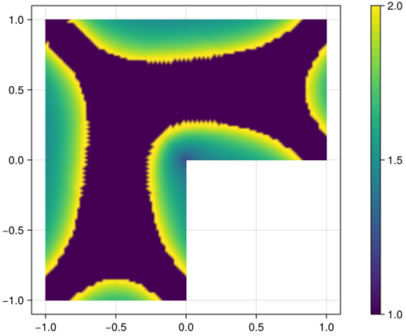

Local contrast (kink with $a_m = 1$, $a_c = 2$)

Locality of constants

Visualization for kink-nonlinearity

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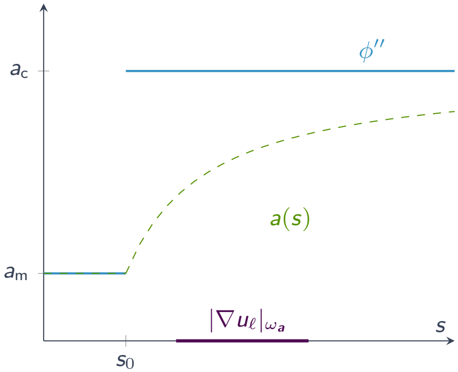

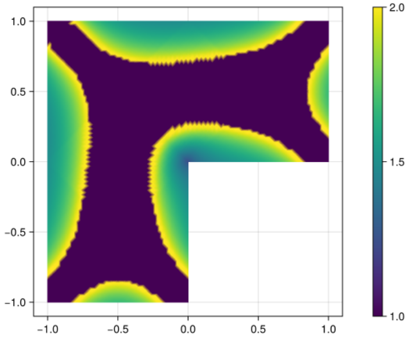

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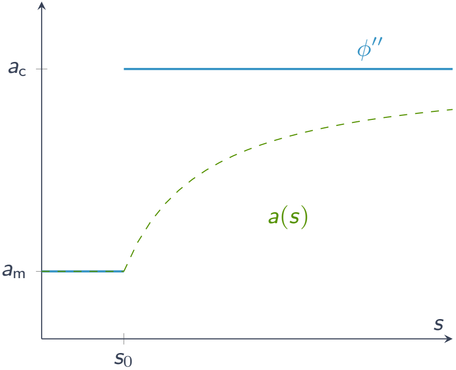
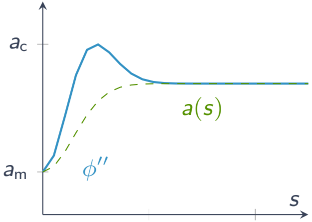


Asymptotically (under refinement):

- $\omega_{a} \to \{a\}$
- $|\nabla \tilde{u}_{\ell+1}| \to |\nabla u_{\ell}|$
- $|A^{-1}(\sigma_{\ell})| \to |\nabla u_{\ell}|$ $(\sigma = -A(\nabla u))$



Exponential nonlinearity

Discrete reliability of the flux estimator
Additional condition for optimality

Theorem (Discrete reliability of $\eta_{\mathrm{flux},\ell}$)

The flux estimator $\eta_{\mathrm{flux},\ell}$ is discretely reliable with local constants, i.e. there holds for an arbitrary refinement $\widehat{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$ that

$$\mathcal{J}(u_\ell) - \mathcal{J}(\hat{u}_\ell) \leq \sum_{\boldsymbol{a} \in \mathcal{V}_\ell^1(\mathcal{S}_\ell)} C_{\mathrm{drel}}^{\boldsymbol{a}} \, \overline{C}_{\mathrm{loc}}^{\boldsymbol{a}} \, \eta_{\mathrm{flux},\ell}^{\boldsymbol{a}}.$$

The constant $C_{\mathrm{drel}}^{\boldsymbol{a}}$ again scales like a patch-local nonlinearity strength.

Remark: Discrete reliability for other estimators from chain of local hierarchy of contributions

A localized lower bound estimator $\underline{\eta}_\ell$

Alternative linear approach

Task: Given the refinement $\mathcal{T}_\ell \to \mathcal{T}_{\ell+1}$, estimate $\mathcal{J}(u_\ell) - \mathcal{J}(u_{\ell+1})$ from below:

$$\mathcal{J}(u_\ell) - \mathcal{J}(u_{\ell+1}) \geq \mathcal{J}(u_\ell) - \mathcal{J}(\tilde{u}_{\ell+1}) \qquad (u_{\ell+1} \text{ minimizer}).$$

$\to$ find lifting $\tilde{u}_{\ell+1} \approx u_{\ell+1}$ of u_ℓ!

Ansatz: Find local residual liftings $r^{\boldsymbol{a}}_{\ell+1} \in V^p_{\ell,\ell+1} \cap H^1_0(\omega_{\boldsymbol{a}})$ such that

$$(\nabla r^{\boldsymbol{a}}_{\ell+1}, \nabla v_{\ell+1})_{\omega_{\boldsymbol{a}}} = (f, v_{\ell+1})_{\omega_{\boldsymbol{a}}} - (\boldsymbol{A}(\nabla u_\ell), \nabla v_{\ell+1})_{\omega_{\boldsymbol{a}}} \qquad \text{for all } v_{\ell+1}$$

- *Linear, homogeneous* local Dirichlet-problem
- We show that there holds

$$\eta^{\boldsymbol{a}}_{\mathrm{energy},\ell} = \mathcal{J}_{\omega_{\boldsymbol{a}}}(u_\ell) - \mathcal{J}_{\omega_{\boldsymbol{a}}}(\tilde{u}^{\boldsymbol{a}}_{\ell+1}) \gtrsim \tfrac{1}{2}\|\nabla r^{\boldsymbol{a}}_{\ell+1}\|^2_{\omega_{\boldsymbol{a}}} =: \eta^{\boldsymbol{a}}_{\mathrm{reslift},\ell}.$$

- **BUT:** no global lifting $\tilde{u}_{\ell+1}$

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Algorithms using one estimator

Reminder: Hierarchy in local contributions

$$\eta^{a}_{\mathrm{flux},\ell} \lesssim \eta^{a}_{\mathrm{res},\ell} \lesssim \eta^{a}_{\mathrm{reslift},\ell} \lesssim \eta^{a}_{\mathrm{energy},\ell}. \tag{16}$$

Observation: All estimators are both upper **and** lower bounds!

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Example 1: Using only $\eta_{\mathrm{flux},\ell}$

- Lower bound implicitly given as

$$\mathcal{J}(u_\ell) - \mathcal{J}(u_{\ell+1}) \geq \sum_{\boldsymbol{a}\in\mathcal{M}_\ell} (d+1)^{-1} \eta^{\boldsymbol{a}}_{\mathrm{energy},\ell} \overset{(16)}{\geq} \sum_{\boldsymbol{a}\in\mathcal{M}_\ell} \underbrace{(d+1)^{-1}\tilde{C}^{\boldsymbol{a}}}_{=:\underline{C}^{\boldsymbol{a}}_{\mathrm{loc}}} \eta^{\boldsymbol{a}}_{\mathrm{flux},\ell}.$$

- Recovers standard *flux-based AFEM* *(similar for residual estimators)*

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Example 2: Using only $\eta_{\mathrm{energy},\ell}$

- If the lower bound is computed for all vertices, it also implicitly yields an upper bound
- Recovers *variational adaptivity* [Heid,Wihler, '18] / *Sobolev gradient flow* [Heid,Houston,Stamm,Wihler, '24]
- Idea of the algorithm: Compute expected energy decrease *everywhere*, standard bulk-chasing criterion and only accept those refinements

Conclusion & outlook

Part II

Discussed estimators:

- Upper bound based on *linear* local flux equilibration
- Lower bound from *nonlinear* local energy minimization
- Residual estimators and residual liftings also fit the framework

Algorithms:

- Two estimators: allows for *adaptive patch refinement* step (avoiding INP)
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- **We prove: All methods contract the energy difference with optimal rate**

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- Upcoming preprint! Also including data oscillation in the flux for $p > 1$
- Estimators for more general convex potentials (e.g. p-Laplacian / ϕ-Laplacian)
  $\rightarrow$ nonlinear localization of the flux equilibration?

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Thank you for your attention!

Data oscillation

Right-hand side f and flux function $\boldsymbol{A}$

Three kinds of oscillation terms occur:

- Classic data oscillation:

$$(\mathrm{id} - \Pi_{p,\ell})(f) \tag{17}$$

  Vanishes for $f \in \mathcal{P}_p(\mathcal{T}_0)$ piecewise polynomial

- Oscillation in the flux:

$$(\mathrm{id} - \boldsymbol{\Pi}^{\mathrm{RT}}_{p-1})(\boldsymbol{A}(\nabla u_\ell)) \tag{18}$$

  Vanishes for polynomial degree $p = 1$

- Discrete oscillations:

$$\Pi_{p,\ell}(\psi^{\boldsymbol{a}} f) - \psi^{\boldsymbol{a}} \Pi_{p\text{-}1,\ell} f, \qquad \Pi_{p,\ell}(\nabla\psi^{\boldsymbol{a}} \cdot \boldsymbol{A}(\nabla u_\ell)) - \nabla\psi^{\boldsymbol{a}} \cdot \boldsymbol{\Pi}^{\mathrm{RT}}_{p\text{-}1,\ell}(\boldsymbol{A}(\nabla u_\ell)) \tag{19}$$

$$(\Pi_{p\text{-}1,\ell} - \Pi_{p,\ell+1})(f - \nabla\cdot\boldsymbol{A}(\nabla u_\ell)), \qquad (\boldsymbol{\Pi}^{\mathrm{RT}}_{p\text{-}1,\ell} - \boldsymbol{\Pi}^{\mathrm{RT}}_{p,\ell+1})(\boldsymbol{A}(\nabla u_\ell)) \tag{20}$$

  Vanish for $f \in \mathcal{P}_{p-1}$ and $p \geq 2$